Universal Point Sets for Planar Graph Drawings with Circular Arcs

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July 22, 2013

Abstract

We prove that there exists a set S of n points in the plane such that every n-vertex planar graph G admits a plane drawing in which every vertex of G is placed on a distinct point of S and every edge of G is drawn as a circular arc.

1 Introduction

It is a classic result of graph theory that every planar graph has a plane *straight-line drawing*, that is, a drawing where vertices are mapped to points in the plane and edges to straight-line segments connecting the corresponding points (achieved independently by Wagner, Fáry, and Stein). Tutte [21] presented the first algorithm, the *barycentric method*, that produces such drawings. Unfortunately, the barycentric method can produce edges whose lengths are exponentially far from each other. Therefore, Rosenstiehl and Tarjan [19] asked whether every planar graph has a plane straight-line drawing where vertices lie on an integer grid of polynomial size. De Fraysseix, Pach, and Pollack [5] and, independently, Schnyder [20] answered this question in the affirmative. Their (very different) methods yield drawings of n-vertex planar graphs on a grid of size $\Theta(n) \times \Theta(n)$, and there are graphs (the so-called "nested triangles") that require this grid size [10]. Later, it was apparently Mohar (according to Pach [6]) who generalized the grid question to the following problem: What is the smallest size $f(n)$ of a *universal point set* for plane straight-line drawings of n-vertex planar graphs, that is, the smallest size (as a function of n) of a point set S such that every n-vertex planar graph G admits a plane straight-line drawing in which the vertices of G are mapped to points in S? The question is listed as problem #45 in the Open Problems Project [6]. Despite more than twenty years of research efforts, the best known lower bound for the value of $f(n)$ is linear in n [4, 17, 18], while the best known upper bound is only quadratic in n, as established by de Fraysseix et al. [5] and Schnyder [20]. Universal point sets for plane straight-line drawings of planar graphs require more than n points whenever $n \geq 15$ [3]. Recently, universal point sets with $o(n^2)$ points have been proved to exist for straight-line planar drawings of several subclasses of planar graphs generalizing outerplanar graphs. Namely, an upper bound of $O(n(\log n/\log\log n)^2$ has been proven for *simply-nested planar graphs* [1] and an upper bound of $O(n^{5/3})$ for *planar 3-trees* [14], which extends to *planar 2-trees* and hence to *series-parallel graphs*.

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Universal point sets have also been studied with respect to different drawing standards. For example, Everett et al. [13] showed that there exist sets of n points that are universal for *plane poly-line drawings with one bend per edge* of n-vertex planar graphs. On the other hand, if bend-points are required to be placed on the point-set, universal point-sets exist of size $O(n^2/\log n)$ for drawings with one bend per edge, of size $O(n \log n)$ for drawings with two bends per edge, and of size $O(n)$ for drawings with three bends per edge [11].

However, smooth curves may be easier for the eye to follow and more aesthetic than poly-lines. Graph Drawing researchers have long observed that poly-lines may be made smooth by replacing each bend with a smooth curve tangent to the two adjacent line segments [7, 15]. Bekos et al. [2] formalized this observation by considering smooth curves made of line segments and circular arcs; they define the *curve complexity* of such a curve to be the number of segments and arcs it contains. A poly-line drawing with s segments per edge may be transformed into a smooth drawing with curve complexity at most $2s - 1$, but Bekos et al. [2] observed that in many cases the curve complexity can be made smaller than this bound. For instance, replacing poly-lines by curves in the construction of Everett et al. [13] would give rise to a drawing of curve complexity 3, but in fact every set of n collinear points is universal for smooth piecewise-circular drawings with curve complexity 2, as can be derived from the existence of topological book embeddings of planar graphs [8, 16, 2]. A *monotone topological book embedding* of a graph G is a drawing of G such that the vertices lie on a horizontal line, called *spine*, and the edges are represented by non-crossing curves, monotonically increasing in the direction of the spine. In [8, 16], it was shown that every planar graph has a monotone topological book embedding where each edge crosses the spine exactly once and consists of two semi-circles, one below and one above the spine (see Figure 2).

The difficulty of the problem of constructing a universal point set of a linear size for straight-line drawings, the aesthetical properties of smooth curves, the recent developments on drawing planar graphs with circular arcs (see, for example, [2, 12]), and the existence of universal sets of n points for drawings of planar graphs with curve complexity 2 [13] naturally give rise to the question of whether there exists a universal set of n points for drawings of planar graphs with curve complexity 1, that is, for drawings in which every edge is drawn as a single circular arc. In this paper, we answer this question in the affirmative.

We prove the existence of set S of n points on the parabolic arc $\mathcal{P} = \{(x, y) \colon x \geq 0, y = -x^2\}$ such that every n-vertex planar graph G can be drawn with the vertices mapped to S and the edges mapped to non-crossing circular arcs. In the same spirit as Everett et al. [13], we draw G in two steps. In the first step, we construct a monotone topological book embedding of G. In the second step, we map the vertices of G to the points in S in the same order as they appear on the spine of the book embedding.

2 Circular Arcs Between Points on a Parabola

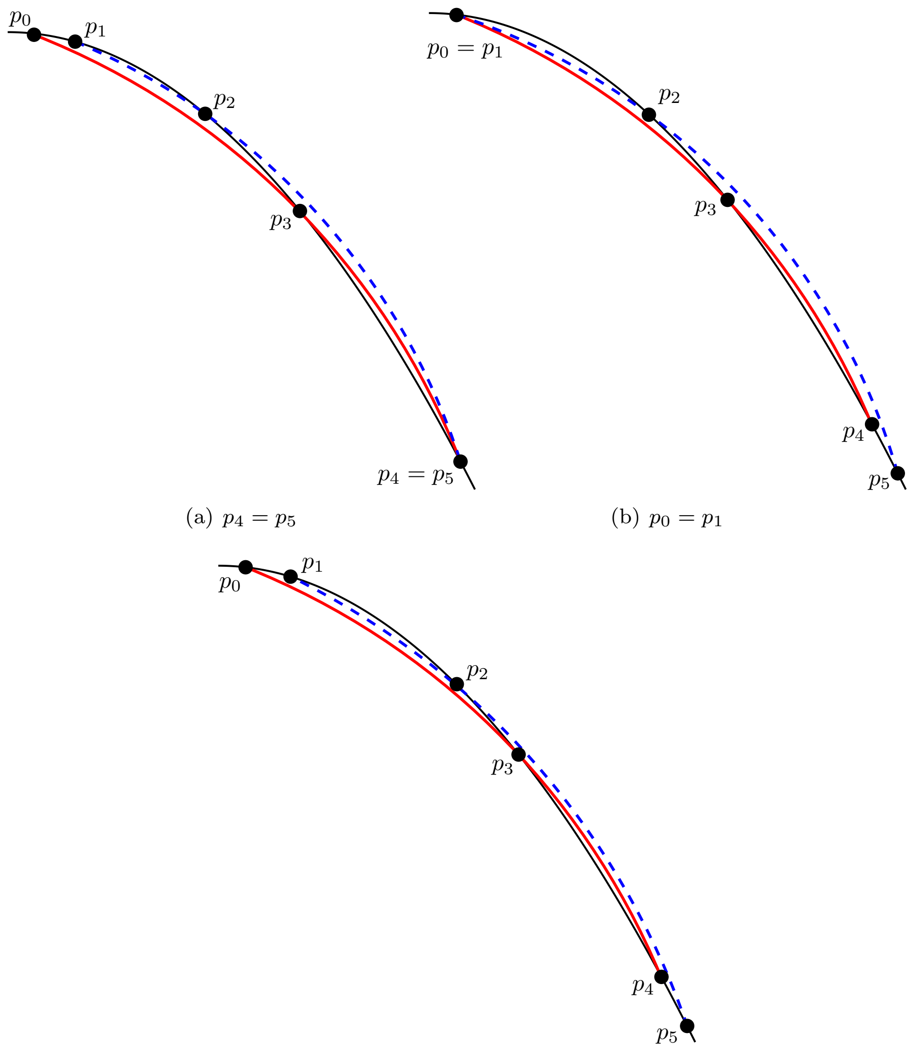
In this section, we investigate geometric properties of circular-arc drawings whose vertices lie on the parabolic arc $\mathcal{P}$.

In the following, when we say that a point is *to the left* of another point, we mean that the x-coordinate of the former is smaller than that of the latter. However, when we say that an arc is to the left of a point q, we mean that all the intersection points of the arc with the horizontal line through q are to the left of q. We define similarly *to the right*, *above*, and *below*, and we naturally extend these definition to non-crossing pairs of arcs. We denote by $\mathcal{C}(p, q, r)$ the circle through three points p, q, and r.

We start by stating a classic property of parabolas and circles.

Lemma 1. *For every three points p, q, and r on $\mathcal{P}$ with increasing x-coordinates, the circular arc from p to r and through q is below $\mathcal{P}$ between p and q and above $\mathcal{P}$ between q and r (see Figure 1).*

Proof. We first observe that a circle intersects $\mathcal{P}$ in at most three points with positive x-coordinates (counted with multiplicity). Indeed, substituting y by $-x^2$ in the circle equation yields a degree-4 equation in x with no monomial of degree 3. There are thus at most three changes of sign in the sequence of coefficients, and Descartes' rule of signs implies that there are at most three positive roots, counted with multiplicity.



(a) $p_4 = p_5$

(b) $p_0 = p_1$

(c) $p_0, \ldots, p_5$ are pairwise distinct

Figure 1: Three configurations of relative position of the circular arcs $C_{0,3,4}$ (red) and $C_{1,2,5}$ (blue dashed) defined by six points $p_0, \ldots, p_5$ lying in that order on $\mathcal{P}$. For readability, the figure is not to scale.

We now consider three points p, q, and r on $\mathcal{P}$ and consider circle $\mathcal{C}(p,q,r)$. Since there is no other point of intersection with positive x-coordinate, and since the circle is bounded and the parabolic arc is not, the circular arc to the right of r is below the parabolic arc. The result follows since $\mathcal{C}(p,q,r)$ crosses $\mathcal{P}$ at p, q, and r (since, otherwise, the number of intersection points with positive x-coordinates and counted with multiplicity would be larger than three). $\square$

Given six points $p_0 = (x_0, y_0), \ldots, p_5 = (x_5, y_5)$ in this order on $\mathcal{P}$ (that is, $x_0 \leq x_1 \leq \cdots \leq x_5$), we consider two circular arcs (see Figure 1); $C_{0,3,4}$ (red) goes through the ordered points p_0, p_3, p_4 and $C_{1,2,5}$ (blue) goes through p_1, p_2, p_5. We assume that the three points defining each arc are pairwise distinct. It should be stressed that these arcs may not be x-monotone.[1] The two circular arcs are, however, y-monotone—for $C_{0,3,4}$ we argue as follows; the argument for $C_{1,2,5}$ is similar: By Lemma 1, p_0 lies on the right half-circle of $\mathcal{C}(p_0, p_3, p_4)$, and p_3 and p_4 are to the right of p_0.

We will prove, in Lemma 4, that the arcs $C_{0,3,4}$ and $C_{1,2,5}$ do not intersect each other if the x-coordinate of p_i is at least twice that of p_{i-1} for $i = 3, 4$. For that purpose, we first consider, in the two next lemmas, the special cases where these arcs share one of their endpoints.

Lemma 2. *If $p_4 = p_5$ and $x_3 \geq x_1 + x_2$, the two circular arcs $C_{0,3,4}$ and $C_{1,2,5}$ intersect only at $p_4 = p_5$.*

Proof. Refer to Figure 1(a). We first observe that, by Lemma 1, the circular arc $C_{0,3,4}$ is below $\mathcal{P}$ in a neighborhood of p_0, it crosses $\mathcal{P}$ at p_3, and it lies above $\mathcal{P}$ in a neighborhood of p_4. Similarly $C_{1,2,5}$ is below $\mathcal{P}$ in a neighborhood of p_1, it crosses $\mathcal{P}$ at p_2, and it lies above $\mathcal{P}$ in a neighborhood of p_5.

We now argue that the two arcs $C_{0,3,4}$ and $C_{1,2,5}$ intersect at a point other than $p_4 = p_5$ if and only if the (red) arc $C_{0,3,4}$ is to the right of the (blue) arc $C_{1,2,5}$ in a neighborhood of p_4. Since the (red) arc $C_{0,3,4}$ is below $\mathcal{P}$ in a neighborhood of p_0, and $C_{0,3,4}$ does not intersect $\mathcal{P}$ between p_0 and p_1 (by Lemma 1), the (red) arc $C_{0,3,4}$ is to the left of p_1. On the other hand, the two circular arcs intersect at most once other than at p_4 (since circles intersect at most twice). Hence, if they intersect at a point q other than p_4, their horizontal ordering changes in a neighborhood of q and thus the (red) arc $C_{0,3,4}$ is to the right of the (blue) arc $C_{1,2,5}$ in a neighborhood of p_4.

As a consequence, we can assume without loss of generality that p_0 is at the origin $O = (0,0)$ (that is, the topmost point of $\mathcal{P}$). This can be seen as follows. First, by Lemma 1, the origin is inside $\mathcal{C}(p_0, p_3, p_4)$. Furthermore, since the origin is above p_3 and p_4, the arc p_3p_4 of $\mathcal{C}(O, p_3, p_4)$ lies to the right of the arc p_3p_4 of $\mathcal{C}(p_0, p_3, p_4)$. It follows that if $C_{0,3,4}$ is to the right of $C_{1,2,5}$ in a neighborhood of p_4, it remains to the right if p_0 is placed at the origin. Hence, in the sequel, we can assume that $x_0 = 0$.

We now prove that if $x_3 \geq x_1 + x_2$, then the tangents at $p_4 = p_5$ of the two circular arcs $C_{0,3,4}$ and $C_{1,2,5}$ are distinct for any position of $p_4 = p_5$ to the right of p_3 on $\mathcal{P}$.

The following calculations are done in Maple. We consider the equation of $\mathcal{C}(p_0, p_3, p_4)$, which is the determinant

$$\begin{bmatrix} x_0 & -x_0^2 & x_0^2 + x_0^4 & 1 \\ x_3 & -x_3^2 & x_3^2 + x_3^4 & 1 \\ x_4 & -x_4^2 & x_4^2 + x_4^4 & 1 \\ x & y & x^2 + y^2 & 1 \end{bmatrix}$$

and similarly for $\mathcal{C}(p_1, p_2, p_4 = p_5)$. The normals to these circles at p_4 are the gradient of their implicit equations evaluated at p_4. We then compute the cross product of these two vectors; more precisely, the last coordinate of the cross product, that is, $M_xN_y - N_xM_y$, where (M_x, M_y) and (N_x, N_y) are the normal vectors.

This expression can be factorized such that it is the product of two terms. The first is the term $x_3\, x_4\, (x_3 - x_4)(x_2 - x_4)(x_1 - x_4)(x_1 - x_2)$, which does not vanish if $p_0, \ldots, p_4$ are pairwise distinct. The second is the

[1] This could be seen by considering, for instance, the limit case of a circle where p_0 and p_3 lie at the origin and the x-coordinate of p_4 is larger than one. This circle is centered at $(0, -a)$ with $a > 1$. Since $-a > -a^2$, the rightmost point $(a, -a)$ of the circle is above the parabola $y = -x^2$, thus it lies on $C_{0,3,4}$ by Lemma 1.

following term, which we view as a polynomial in x_4 whose coefficients depend on x_1, x_2, and x_3:

$$
\begin{aligned}
&(x_3 - x_1 - x_2)\, x_4^4 \\
+&(x_1 + x_2 + x_3)\,(x_3 - x_1 - x_2)\, x_4^3 \\
+&(1 + x_1 x_2)\,(x_3 - x_1 - x_2)\, x_4^2 \\
+&(x_1 x_2 x_3^2 + x_1 x_2^2 x_3 + x_1^2 x_2 x_3 + x_3^2 - x_1^2 - x_2^2)\, x_4 \\
+&x_1\, x_2\,(1 + x_3^2)\,(x_1 + x_2).
\end{aligned}
$$

All coefficients are non-negative since $x_3 \geq x_1 + x_2$. Thus, the polynomial has no positive real root. In other words, the two normals are never collinear. Now, considering the limit case where $p_4 = p_3$, the (red) circle $\mathcal{C}(p_0, p_3, p_4)$ is tangent to $\mathcal{P}$ and since, by Lemma 1, the (blue) arc $C_{1,2,5}$ is above and thus to the right of $\mathcal{P}$ in a neighborhood of $p_4 = p_5$ (and is not tangent to $\mathcal{P}$ if $p_2 \neq p_5$), the (blue) arc $C_{1,2,5}$ is to the right of the (red) arc $C_{0,3,4}$ in a neighborhood of p_4. Hence, the two arcs $C_{0,3,4}$ and $C_{1,2,5}$ do not intersect except at p_4. □

Lemma 3. *If $p_0 = p_1$, $x_0 \geq 1$, $x_3 \geq 2x_2$ and $x_4 \geq x_0 + x_3$, the two circular arcs $C_{0,3,4}$ and $C_{1,2,5}$ intersect only at $p_0 = p_1$.*

Proof. Similarly as in the proof of Lemma 2, the two arcs $C_{0,3,4}$ and $C_{1,2,5}$ intersect at a point other than $p_0 = p_1$ if and only if the (red) arc $C_{0,3,4}$ is to the right of the (blue) arc $C_{1,2,5}$ in a neighborhood of p_0 (see Figure 1(b)).

Furthermore, we can assume without loss of generality that p_5 is at infinity, which means that $C_{1,2,5}$ is the (straight) ray from $p_0 = p_1$ through p_2. Indeed, for any point p_5' that lies on $\mathcal{P}$ to the right of p_5, point p_5' lies outside the $\mathcal{C}(p_1, p_2, p_5)$ by Lemma 1. Furthermore, since p_5' lies below p_1 and p_2, the arc through p_1, p_2, and p_5' (in order) lies to the left of $C_{1,2,5}$ between p_1 and p_2. Hence, if the (blue) arc $C_{1,2,5}$ is to the left of the (red) arc $C_{0,3,4}$ in a neighborhood of p_0, it remains to the left if p_5 is at infinity.

Now, similarly to the proof of Lemma 2, we prove that the tangents at $p_0 = p_1$ of $C_{0,3,4}$ and $C_{1,2,5}$ never coincide. With the above assumption, this is equivalent to showing that the normal to $C_{0,3,4}$ at p_0 is never orthogonal to the segment p_1p_2. The corresponding dot product (computed in Maple) is equal to

$$
\begin{aligned}
&(x_4 - x_3)\,(x_4 - x_0)\,(x_3 - x_0)\,(x_2 - x_0) \\
&\Big((x_3 - x_2)\, x_4^2 + (x_3 - x_2)\,(x_0 + x_3)\, x_4 + \\
&\quad \big((x_0^2 - 1 - x_3 x_0 - x_3^2)\, x_2 + x_0^3 + x_0\big)\Big).
\end{aligned}
$$

The first four terms never vanish and we want to show that the last term, seen as a polynomial in x_4, has no root x_4 larger than $x_0 + x_3$ (it can be shown that this polynomial has a positive root). For that purpose, we make the change of variable $x_4 = t + x_0 + x_3$ which maps the interval $(x_0 + x_3, +\infty)$ of x_4 to the interval $(0, +\infty)$ of t and maps the above degree-2 polynomial in x_4 to

$$
\begin{aligned}
&(x_3 - x_2)\, t^2 + 3\,(x_3 - x_2)\,(x_0 + x_3)\, t - \\
&(1 + x_0^2 - 5x_0 x_3 + 3x_3^2)\, x_2 + \\
&\quad x_0 + 4x_0 x_3^2 + x_0^3 + 2x_3^3 + 2x_0^2 x_3
\end{aligned}
$$

whose first and second coefficients are positive and whose last coefficient is positive for any $x_2 \in [x_0, x_3/2]$ since it is linear in x_2 and takes value $x_3\,(3x_0 + 2x_3)\,(x_3 - x_0)$ at x_0 and value $\frac{1}{2} x_3\,(-1 + x_3^2 + 3x_0^2 + 3x_0 x_3) + x_0 + x_0^3$ at $x_3/2$ (which is positive since $x_0 \geq 1$).[2] Hence, if $x_3 \geq 2x_2$, all coefficients of this polynomial are positive, which implies that it has no positive roots. This, in turn, means that the initial degree-2 polynomial in x_4 has no root larger than $x_0 + x_3$.

[2] Note that the last coefficient is negative when $x_2 = x_3$ which is why we consider x_2 in the range $[x_0, x_3/2]$.

This implies that there is no position of the points $p_0 = p_1, p_2 \ldots, p_5$ such that $x_3 \geq 2x_2$, $x_4 \geq x_0 + x_3$ and such that the tangent to $C_{0,3,4}$ is collinear with p_0p_2. Furthermore, at the limit case where $p_2 = p_0$, the segment p_0p_2 is tangent to $\mathcal{P}$, and $C_{0,3,4}$ is below and to the left of that tangent in a neighborhood of p_0 (by Lemma 1). Hence, for any position of the points $p_0 = p_1, p_2 \ldots, p_5$ (as defined above) such that $x_3 \geq 2x_2$, $x_4 \geq x_0 + x_3$, the (red) circular arc $C_{0,3,4}$ is to the left of the segment p_1p_2 in a neighborhood of p_0. Finally, as argued above when we considered p_5 at infinity, this implies that for any position of the points $p_0 = p_1, p_2, \ldots, p_5$ such that $x_3 \geq 2x_2$ and $x_4 \geq x_0 + x_3$, the (red) circular arc $C_{0,3,4}$ is to the left of the (blue) circular arc $C_{1,2,5}$ in a neighborhood of $p_0 = p_1$. This concludes the proof since we have proved that this is equivalent to the property that the arcs $C_{0,3,4}$ and $C_{1,2,5}$ intersect only at $p_0 = p_1$. □

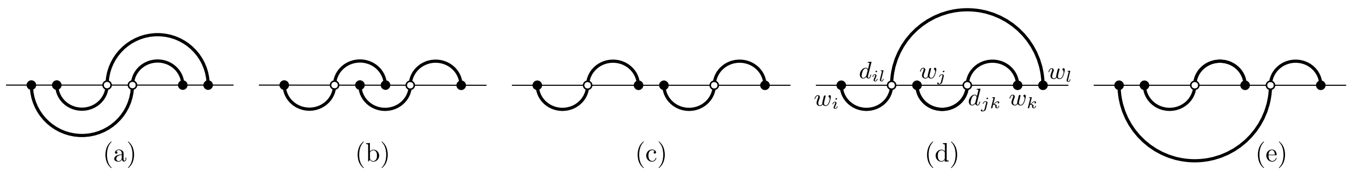


Figure 2: Relative positions of two edges in a monotone topological book embedding.

Lemma 4. *If $p_0, \ldots, p_5$ are pairwise disjoint and $x_i \geq 2x_{i-1}$ for $i = 3, 4$, the two circular arcs $C_{0,3,4}$ and $C_{1,2,5}$ do not intersect.*

Proof. We refer to Figure 1(c) and, unless specified otherwise, an arc p_ip_j refers to the arc from p_i to p_j on the arc $C_{0,3,4}$ or $C_{1,2,5}$ that supports both p_i and p_j. We first prove that the arcs p_2p_5 and p_3p_4 do not intersect. For any point q on $\mathcal{P}$ between p_4 and p_5, the arc p_3q on the circular arc through p_0, p_3, q lies above the concatenation of the arcs p_3p_4 of $C_{0,3,4}$ and p_4q of $\mathcal{P}$ (since the circular arcs p_3q and p_3p_4 lie above $\mathcal{P}$, by Lemma 1, and $\mathcal{C}(p_0, p_3, p_4)$ and $\mathcal{C}(p_0, p_3, q)$ intersect only at p_0 and p_3). It follows that if arc p_3p_4 intersects arc p_2p_5, then arc p_3q also intersects arc p_2p_5 for any position of q between p_4 and p_5 on $\mathcal{P}$. This implies that, for the limit case where $q = p_5$, arc $C_{1,2,5}$ and the circular arc through p_0, p_3, and $q = p_5$ intersect in some point other than $q = p_5$, which is not the case by Lemma 2.

We now prove, similarly, that the arcs p_0p_3 and p_1p_2 do not intersect. For any point q on $\mathcal{P}$ between p_0 and p_1, the arc qp_2 on the circular arc through q, p_2, p_5 lies below the concatenation of the arcs qp_1 of $\mathcal{P}$ and p_1p_2 of $C_{1,2,5}$. It follows that if arc p_1p_2 intersects arc p_0p_3, then arc qp_2 also intersects arc p_0p_3 for any position of q between p_0 and p_1 on $\mathcal{P}$. This implies that, for the limit case where $q = p_0$, arc $C_{0,3,4}$ and the circular arc through $q = p_0$, p_2, and p_5 intersect in some point other than $q = p_0$, which is not the case by Lemma 3.

Finally, arcs p_1p_2 of $C_{1,2,5}$ and p_3p_4 of $C_{0,3,4}$ do not intersect because they lie on different sides of $\mathcal{P}$ and similarly for arcs p_0p_3 of $C_{0,3,4}$ and p_2p_5 of $C_{1,2,5}$. Hence, the two arcs $C_{0,3,4}$ or $C_{1,2,5}$ do not intersect. □

3 Universal Point Set for Circular Arc Drawings

In this section, we construct a set of n points on $\mathcal{P}$ and, by using the lemmata of the previous section, we prove that it is universal for plane circular arc drawings of n-vertex planar graphs.

Consider n^2 points $q_0, \ldots, q_{n^2-1}$ on the parabolic arc $\mathcal{P}$ such that $x_0 \geq 1$ and $x_i \geq 2x_{i-1}$ for $i = 1, \ldots, n^2 - 1$. For our universal point set, we take the n points $p_i = q_{ni}$ for $i = 0, \ldots, n-1$. We call the points in $q_0, \ldots, q_{n^2-1}$ that are not in the universal point set *helper points.*

Theorem 5. *Every n-vertex planar graph can be drawn with the vertices on $p_0, \ldots, p_{n-1}$ and circular edges that do not intersect except at common endpoints.*

Proof. Consider any planar graph G. Construct a monotone topological book embedding Γ of G in which all edges are drawn with a spine crossing [8, 16]. Denote by $w_0, \ldots, w_{n-1}$ the order of the vertices of G on the spine in Γ. We substitute every spine crossing with a *dummy* vertex. The relative position of any two edges in Γ is as depicted in Figure 2 (in which two edges may share their endpoints). For $0 \leq i \leq n-1$, we map vertex w_i to point p_i. Furthermore, for each $0 \leq i \leq n-2$, we map the dummy vertices that lie in between w_i and w_{i+1} on the spine in Γ to distinct helper points in between p_i and p_{i+1}, so that the order of the dummy vertices on $\mathcal{P}$ is the same as on the spine in Γ. (We postpone the proof that there are enough points q_i to map the dummy vertices.) We finally draw every edge (w_i, w_j) of G containing a dummy vertex d_l as a circular arc passing through p_i, through p_j, and through the helper point to which vertex d_l has been mapped to. We prove that the resulting drawing is plane.

By Lemmata 2, 3, and 4, two edges whose relative positions in Γ are as depicted in Figure 2(a) do not intersect except possibly at a common endpoint.

For the pairs of edges whose relative positions in Γ are as depicted in Figures 2(b) and 2(c), it is straightforward to check that they do not intersect either because they are separated by $\mathcal{P}$, or because they are y-monotone and hence they are separated by a horizontal line.

Consider two edges (w_i, w_l) and (w_j, w_k) whose relative position in Γ is as depicted in Figure 2(d) (the argument for pairs of edges as in Figure 2(e) is analogous). Let d_{il} and d_{jk} be the dummy vertices of (w_i, w_l) and (w_j, w_k), respectively. Let q_{il} and q_{jk} be the points on $\mathcal{P}$ to which d_{il} and d_{jk} are mapped. Arcs $p_i q_{il}$ and $p_j p_k$ do not intersect because they are both y-monotone and their endpoints are separated by a horizontal line. Arcs $q_{il} p_l$ and $p_j q_{jk}$ do not intersect because they are separated by $\mathcal{P}$. Hence, it suffices to prove that arcs $q_{jk} p_k$ and $q_{il} p_l$ do not intersect. These two arcs are above and to the right of $\mathcal{P}$ (by Lemma 1) and q_{il}, q_{jk}, p_k, p_l are ordered from top to bottom. It is thus sufficient to prove that there exists a curve from q_{jk} to p_k that is to the right of $q_{jk} p_k$ and that does not intersect $q_{il} p_l$. Consider the (y-monotone) arc from q_{jk} to p_k of the circle $\mathcal{C}(p_i, q_{jk}, p_k)$. It is indeed to the right of the arc $q_{jk} p_k$ (of $\mathcal{C}(p_j, q_{jk}, p_k)$) because p_i is inside $\mathcal{C}(p_j, q_{jk}, p_k)$ (by Lemma 1) and p_i, q_{jk}, and p_k are ordered on the parabola. Furthermore, this new arc does not intersect $q_{il} p_l$ because in the case where $w_i = w_j$, w_k and w_l are in this order on the spine—that's the situation depicted in Figure 2(a)—we know that the corresponding circular arcs do not intersect.

It remains to show that there are enough helper points to map the dummy vertices. There are $n-1$ helper points $q_{ni+1}, \dots, q_{n(i+1)-1}$ between each pair of points $p_i = q_{ni}$ and $p_{i+1} = q_{n(i+1)}$. It thus suffices to prove that there are at most $n-1$ dummy vertices in between w_i and w_{i+1} along the spine in Γ.

Let $(u_1, v_1), \dots, (u_k, v_k)$ be k edges in the book embedding that define consecutive dummy vertices on the spine. If no vertex w_i lies in between these dummy vertices on the spine in Γ, the k edges are such that $u_1, \dots, u_k, v_1, \dots, v_k$ are ordered from left to right on the spine in Γ; see Figure 3(a). Now, consider the graph that consists of these edges plus the edges $(u_i, u_{i+1}), (v_i, v_{i+1})$, for $i = 1, \dots, k-1$; see Figure 3(b). This graph is outerplanar. It has at most n vertices and, thus, at most $n-3$ chords. On the other hand, it has exactly $k-2$ chords: $(u_2, v_2), \dots, (u_{k-1}, v_{k-1})$. This implies that $k-2 \leq n-3$ and $k \leq n-1$, which concludes the proof. □

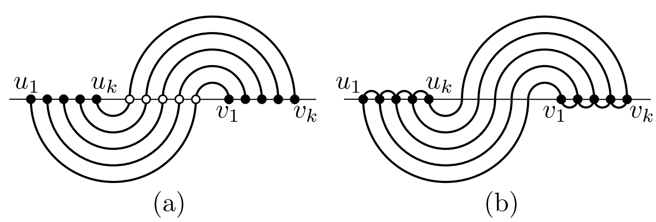


Figure 3: (a) k edges of a monotone topological book embedding that defines k consecutive dummy vertices (spine crossings). (b) Augmented outerplanar graph.

4 Conclusions

We proved the existence of a universal point set with n points for plane circular arc drawings of planar graphs. The universal point set we constructed has an area of $2^{O(n^2)}$. It would be interesting, also for practical visualization purposes, to construct a universal point set with n points for plane circular arc drawings of planar graphs within polynomial area. We remark that (relaxing the requirement that the set have exactly n points) a universal point set with $O(n)$ points and within $2^{O(n)}$ area for plane circular arc drawings of planar graphs is $Q = \{q_0, \ldots, q_{4n-7}\}$, where the helper points are defined as in Section 3. To construct a plane circular-arc drawing of a planar graph G on Q, it suffices to map vertices and dummy vertices of a monotone topological book embedding of G to the points of Q in the order they appear in the book embedding. The geometric lemmata of Section 2 ensure that the resulting drawing is plane.

Acknowledgments

This research started during Dagstuhl Seminar 13151 "Drawing Graphs and Maps with Curves" in April 2013. The authors thank the organizers and the participants for many useful discussions.

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